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BOUNDS AND APPROXIMATIONS FOR THE RENEWAL FUNCTION

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THESIS

BOUNDS AND APPROXIMATIONS FOR
THE RENEWAL FUNCTION

by

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Bounds and Approximations for

the Renewal Function

by

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ABSTRACT

This thesis gives a number of approximations and bounds for the renewal function in an ordinary renewal process. Each approximation and bound is calculated for the uniform, gamma and hyperexponential distributions and compared with the renewal function for these cases. They are also calculated for the log-normal distribution and compared with results of the simulation of the renewal function. Results are tabulated and shown graphically.

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I. INTRODUCTION

Let $\{X_1, X_2, \ldots\}$ be a sequence of independent identically distributed non-negative random variables. Let the distribution function of X_i, $i = 1, 2, \ldots$ be $F(x)$. Let

$$S_n = X_1 + X_2 + \quad + X_n, \quad n \geq 1,$$

the n^{th} partial sum, and $S_o = 0$. Define the random variable

$$N(t) = \text{Sup} \{ n \mid S_n \leq t \},$$

which is the number of renewals occurring before and including t. The renewal function, $M(t)$, is defined as

$$M(t) = E(N(t)) , \quad t \geq 0. \tag{1}$$

Let $F_n(t)$ be the n-fold convolution of F with itself. The relationship between $M(t)$ and $F(t)$ is given by the following,

$$M(t) = \sum_{n=1}^{\infty} F_n(t),$$

which leads to the well-known renewal equation,

$$M(t) = F(t) + \int_o^t M(t - x) \, dF(x). \tag{2}$$

The purpose of this thesis is to investigate approximations and bounds for the renewal function, since Equations (1) and (2) can only be solved for special cases of the distribution function F.

In Section II formulae are collected for various approximations and bounds that have appeared in the literature (Barlow & Proschan (1965), Bartholomew (1963), and Butterworth and Marshall (1971)). These formulae are applied to the uniform, gamma, hyperexponential and lognormal distributions. In Section III the results of computations are presented in both tabular and graphical forms for each approximation applied to each distribution.

II. APPROXIMATION FORMULAE

Throughout this thesis the following notation will be used:

$$E[X_i] = \frac{1}{\lambda} ,$$

$$k = \frac{\lambda^2 E[X_i^2]}{2} - 1,$$

and

$$F_e(t) = \lambda \int_0^t (1 - F(u))\, du , \quad t \geq 0.$$

With this notation F_e is an equilibrium excess distribution of a renewal process, and from Smith (1958), if F is non-lattice,

$$M(t) = \lambda t + k + o(1). \tag{3}$$

This equation, together with (2), plays an important part in the development of approximations to M(t).

1. A Simple Lower Bound

Butterworth and Marshall (1971) show that for any renewal process

$$M(t) \geq \lambda t + \sum_{k=1}^{n} F_k(t) - \sum_{k=1}^{n} F_e * F_{k-1}(t) - F_n(t), \quad n \geq 0, \tag{4}$$

where $F_0(t) = 1$, and when n=0 both summations are assumed to be empty. The right-hand side of (4) gives an increasing sequence of lower bounds which converge monotonically on M(t). The case n=1 will be used in this thesis and we define

$$A(t) = \lambda t - F_e(t) , \qquad t \geq 0. \tag{5}$$

2. <u>An Improved Lower Bound</u>

The following general lower bound was found by Barlow and Proschan (1965). Define

$$B(t) = \frac{\lambda t}{F_e(t)} - 1. \tag{6}$$

Then $B(t) \leq M(t)$. Since $B(t) = (A(t)/F_e(t))$, it is clear that B is an improvement over A when $F_e(t) < 1$.

3. <u>Upper Bounds When F is NBUE</u>

A distribution of a non-negative random variable is called NBUE (new better than used in expectation) if

$$\int_t^\infty \frac{(1-F(x))}{1-F(t)} \, dx \leq \frac{1}{\lambda} , \tag{7}$$

for all t where $F(t) < 1$. Equivalently F is NBUE iff $F_e(t) \geq F(t)$ for all t. For details concerning this and other classes of distributions frequently used in reliability theory see Marshall and Proschan (1970). Butterworth and Marshall (1971) show that if F is NBUE,

$$M(t) \leq \lambda t + \sum_{k=1}^{n} F_k(t) - \sum_{k=1}^{n} F_e * F_{k-1}(t) \quad , \; n \geq 0, \tag{8}$$

where again $F_0(t) = 1$, and when n=0, both summations are taken to be empty. The right-hand-side of (6) gives a decreasing sequence of upper bounds which converge monotonically to M(t).[1] The case n=2 will be used in this thesis.

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[1] If the inequality is reversed in (7), F is called NWUE (new worse than used in expectation). In this case (8) gives an improved sequence of lower bounds over the sequence given by (4).

We define

$$C(t) = \lambda t + F(t) - F_e(t) \tag{9}$$

$$= A(t) + F(t).$$

4. Upper Bound When F is IFR

A distribution is said to have increasing failure rate (is IFR) if Log(1-F(t)) is concave in t. Under this assumption Barlow & Proschan (1965) showed that if we define

$$D(t) = \frac{\lambda t F(t)}{F_e(t)}, \tag{10}$$

then $D(t) \geq M(t)$.

5. Approximations for M(t)

Bartholomew (1963) derived the following approximation for the renewal density $m(t) = (dM(t)/dt)$ when F has a density f;

$$m(t) \simeq f(t) + \frac{\lambda\ F(t)^2}{F_e(t)} .$$

The following integral of this approximation is used as an approximation for M(t),

$$E(t) = F(t) + \lambda \int_0^t \frac{F(u)^2}{F_e(u)}\, du. \tag{11}$$

Another form of the renewal function can be derived from (1) or (2). One can show that

$$M(t) = \lambda t + \int_0^t [1 - F_e(t-u)]\ dM(u) - F_e(t). \tag{12}$$

If we approximate dM(u) on the right hand side by λdu we get another approximation for M. Thus define,

$$G(t) = \lambda t + \lambda \int_0^t [1 - F_e(u)]\, du - F_e(t). \qquad (13)$$

Recall that A(t) and B(t) gave simple lower and upper bounds on M for F restricted to NBUE. However, $(A(t)-\lambda t)\to -1$ and $(B(t)-\lambda t)\to 0$ as $t\to 0$. From (3) for F non-lattice it is known that $(M(t)-\lambda t)\to k$, so define

$$H(t) = \lambda t - F_e(t) + (1+k)\, F(t), \qquad (14)$$

the convex combination of A and B with the correct asymptotic behavior.

6. <u>Distribution Tested</u>

The following distributions with specified parameters are used to compare the bounds and approximations for M(t).

a. Uniform Distribution ($\lambda=1$)

$$F(t) = \frac{t}{2} \qquad \text{if } 0 \le t < 2$$

$$= 1 \qquad \text{if } 2 \le t.$$

Then

$$F_e(t) = t - \frac{t^2}{4} \qquad \text{if } 0 \le t < 2$$

$$= 1 \qquad \text{if } 2 \le t.$$

These give:

$$A(t) = \frac{t^2}{4} \qquad \text{if } 0 \le t < 2$$

$$= t - 1 \qquad \text{if } 0 \le t,$$

$$B(t) = \frac{t}{4-t} \qquad \text{if } 0 \le t < 2$$

$$= t - 1 \qquad \text{if } 2 \le t,$$

$$C(t) = \frac{t^2}{4} + \frac{t}{2} \qquad \text{if } 0 \le t < 2$$

$$= t \qquad \text{if } 2 \le t,$$

$$D(t) = \frac{2t}{4-t} \qquad \text{if } 0 \le t < 2$$

$$= t \qquad \text{if } 2 \le t,$$

$$E(t) = 4\log\left(\frac{4}{4-t}\right) - \frac{t}{2} \qquad \text{if } 0 \le t < 2$$

$$= t - 4\log\left(\frac{1}{2}\right) - 3 \qquad \text{if } 2 \le t,$$

$$G(t) = \frac{t^3}{12} - \frac{t^2}{4} + t \qquad \text{if } 0 \le t < 2$$

$$= t - \frac{1}{3} \qquad \text{if } 2 \le t,$$

$$H(t) = \frac{t^2}{4} + \frac{t}{3} \qquad \text{if } 0 \le t < 2$$

$$= t - \frac{1}{3} \qquad \text{if } 2 \le t.$$

For this case,

$$M(t) = \left[\sum_{n=0}^{j-1} \frac{1}{n!} \left(n-\frac{t}{2}\right)^n e^{\frac{t}{2}-n}\right] - 1 \quad \text{if } (j-1)(2) \le t \le (j)(2)$$
$$j = 1,2,\ldots \quad .$$

b. Gamma Distribution (λ=1)

$$F(t) = 1 - (1 + 2t)\, e^{-2t},$$

$$F_e(t) = 1 - (1 + t)\, e^{-2t}.$$

These give:

$$A(t) = t + (1 + t)\, e^{-2t} - 1,$$

$$B(t) = \frac{t}{1 - (1 + t)\, e^{-2t}} - 1,$$

$$C(t) = t - t\, e^{-2t},$$

$$D(t) = \frac{t - t\, e^{-2t} - 2\, t^2\, e^{-2t}}{1 - e^{-2t} - t\, e^{-2t}},$$

$$E(t) = 1 - e^{-2t} - 2\, t\, e^{-2t} + \int_0^t \frac{(1 - e^{-2u} - 2u\, e^{-2u})^2}{1 - e^{-2u} - u\, e^{-2u}}\, du,$$

$$G(t) = t + \left(\frac{1}{2}\, t + \frac{1}{4} \right) e^{-2t} - \frac{1}{4},$$

$$H(t) = t - \frac{1}{2} \left(t - \frac{1}{2} \right) e^{-2t} - \frac{1}{4}.$$

For this case

$$M(t) = t + \frac{1}{4}\, e^{-4t} - \frac{1}{4}.$$

c. Hyperexponential Distribution (λ=1)

$$F(t) = \frac{3}{4} \left(1 - e^{-2t} \right) + \frac{1}{4} \left(1 - e^{-\frac{2}{5} t} \right),$$

$$F_e(t) = \frac{3}{8} \left(1 - e^{-2t} \right) + \frac{5}{8} \left(1 - e^{-\frac{2}{5} t} \right).$$

These give:

$$A(t) = t + \frac{3}{8}\, e^{-2t} + \frac{5}{8}\, e^{-\frac{2}{5} t} - 1,$$

$$B(t) = \frac{8t}{3(1 - e^{-2t}) + 5(1 - e^{-\frac{2}{5} t})} - 1$$

$$C(t) = t + \frac{3}{8} \left(1 - e^{-2t} \right) - \frac{3}{8} \left(1 - e^{-\frac{2}{5} t} \right),$$

$$D(t) = \frac{2t\left[3(1 - e^{-2t}) + (1 - e^{-\frac{2}{5} t})\right]}{3(1 - e^{-2t}) + 5(1 - e^{-\frac{2}{5} t})},$$

$$E(t) = \frac{3}{4}\left(1 - e^{-2t}\right) + \frac{1}{4}\left(1 - e^{-\frac{2}{5}t}\right)$$

$$+ \int_0^t \frac{[3(1 - e^{-2u}) + (1 - e^{-\frac{2}{5}u})]^2}{6(1 - e^{-2u}) + 10(1 - e^{-\frac{2}{5}u})}\, du ,$$

$$G(t) = t - \frac{3}{16}\left(1 - e^{-2t}\right) + \frac{15}{16}\left(1 - e^{-\frac{2}{5}t}\right),$$

$$H(t) = t + \frac{15}{16}\left(1 - e^{-2t}\right) - \frac{3}{16}\left(1 - e^{-\frac{2}{5}t}\right).$$

For this case

$$M(t) = t + \frac{3}{4}\left(1 - e^{-\frac{4}{5}t}\right).$$

d. Log-normal Distribution ($\lambda = \sqrt{e}$)

Let

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{X^2}{2}}\, dx .$$

Then

$$F(t) = \Phi(\log t)$$

$$F_e(t) = te^{-\frac{1}{2}}(1 - \Phi(\log t)) + \Phi((\log t) - 1).$$

These give:

$$A(t) = e^{-\frac{1}{2}} t\, \Phi(\log t) - \Phi((\log t) - 1),$$

$$B(t) = \frac{e^{-\frac{1}{2}} t}{e^{-\frac{1}{2}} t(1 - \Phi(\log t)) + \Phi((\log t) - 1)} - 1,$$

$$C(t) = (e^{-\frac{1}{2}} t + 1)\ \Phi(\log t) - \Phi((\log t) - 1),$$

$$D(t) = \frac{e^{-\frac{1}{2}} t\ \Phi(\log t)}{e^{-\frac{1}{2}} t(1 - \Phi(\log t)) + \Phi((\log t) - 1)},$$

$$E(t) = \Phi(\log t) + e^{-\frac{1}{2}} \int_{0}^{t} \frac{(\Phi(\log u))^2\ du}{e^{-\frac{1}{2}} u(1 - \Phi(\log u)) + \Phi((\log u) - 1)}$$

$$\begin{aligned} G(t) = {} & e^{-\frac{1}{2}} t - \frac{e}{2} (1 - \Phi((\log t) - 2)) \\ & + (e^{-\frac{1}{2}} t + 1)\ (1 - \Phi((\log t) - 1)) \\ & - \frac{1}{2} e^{-\frac{1}{2}} t\ (e^{-\frac{1}{2}} t + 2)(1 - \Phi(\log t)) + \frac{e}{2} - 1, \end{aligned}$$

$$H(t) = (e^{-\frac{1}{2}} t + \frac{1}{2} e)\ \Phi(\log t) - \Phi((\log t) - 1).$$

For this case the exact form of $M(t)$ cannot be found.

III. NUMERICAL RESULTS

Numerical calculations are presented for each bound and approximation and all four distributions in Section II. Results are tabulated below and are graphed and comparisons are made with M(t). For the log-normal case M(t) was simulated for comparison with approximations.

T	A(T)	B(T)	C(T)	D(T)	E(T)	G(T)	H(T)	M(T)
0.1	0.002	0.026	0.052	0.051	0.051	0.098	0.036	0.051
0.2	0.010	0.053	0.110	0.105	0.105	0.191	0.077	0.105
0.3	0.022	0.081	0.172	0.162	0.162	0.280	0.122	0.162
0.4	0.040	0.111	0.240	0.222	0.221	0.365	0.173	0.221
0.5	0.062	0.143	0.312	0.286	0.284	0.448	0.229	0.284
0.6	0.090	0.176	0.390	0.353	0.350	0.528	0.290	0.350
0.7	0.122	0.212	0.472	0.424	0.419	0.606	0.356	0.419
0.8	0.160	0.250	0.560	0.500	0.493	0.683	0.427	0.492
0.9	0.202	0.290	0.652	0.581	0.570	0.758	0.502	0.568
1.0	0.250	0.333	0.750	0.667	0.651	0.833	0.583	0.649
2.0	1.000	1.000	2.000	2.000	1.773	1.667	1.667	1.718
5.0	4.000	4.000	5.000	5.000	4.773	4.667	4.667	4.666
8.0	7.000	7.000	8.000	8.000	7.773	7.667	7.667	7.666
10.0	9.000	9.000	10.000	10.000	9.773	9.667	9.667	9.666

TABLE I. M(t) and approximations for the uniform distribution with mean 1 and variance 1/3.

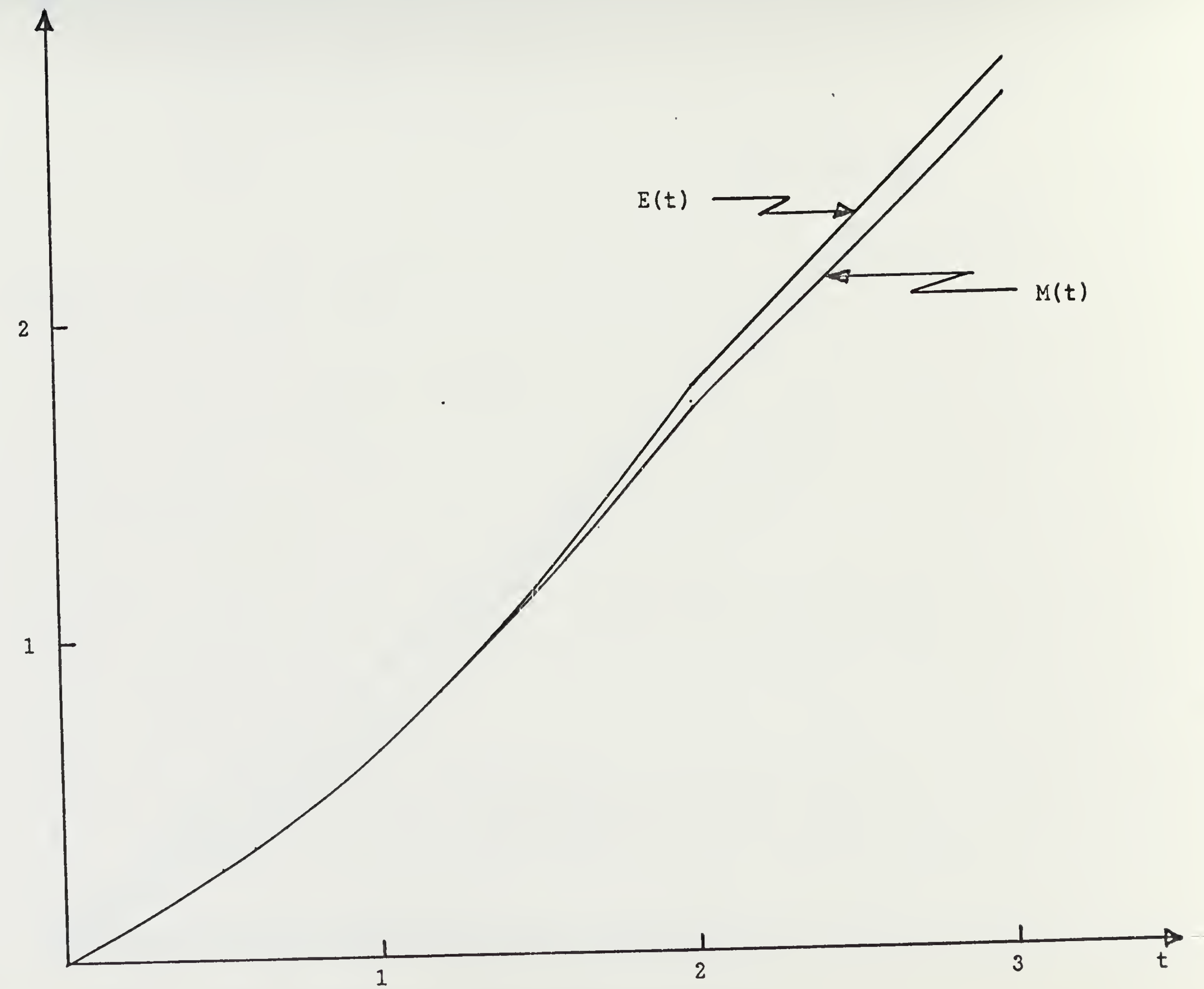

FIGURE 1 : M(t) and E(t) for the uniform distribution.

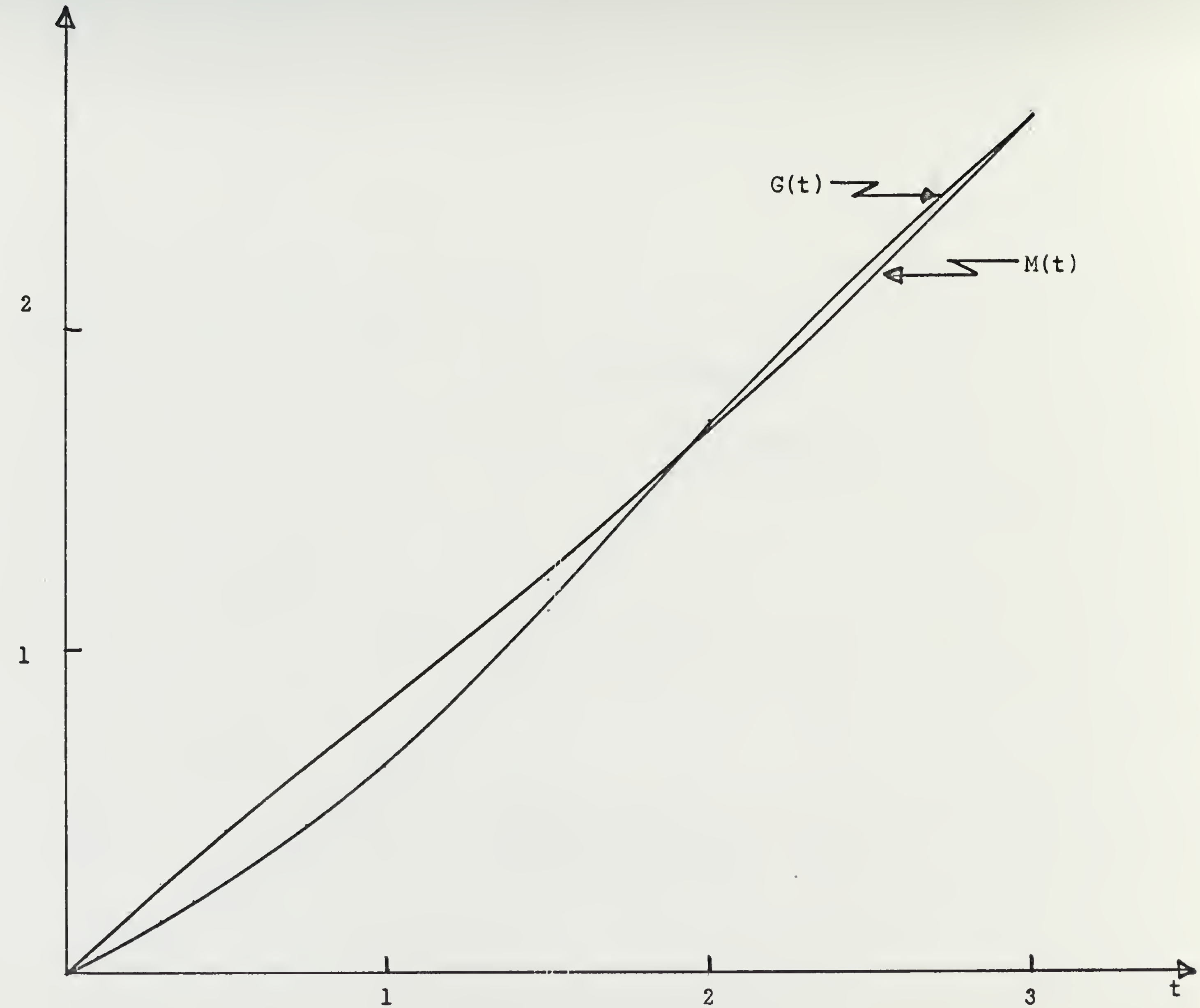

FIGURE 2 : M(t) and G(t) for the uniform distribution.

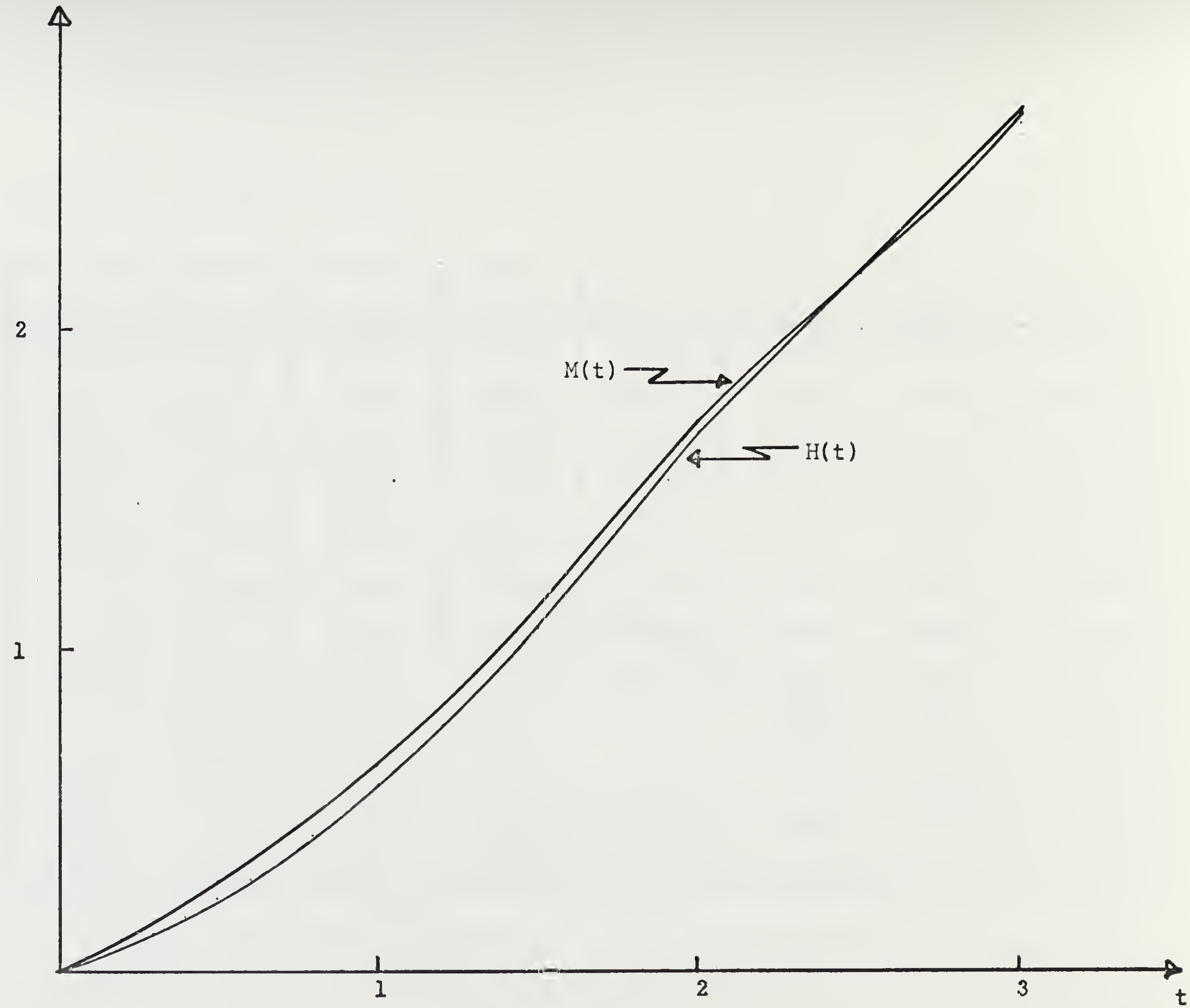

FIGURE 3 : M(t) and H(t) for the uniform distribution.

T	A(T)	B(T)	C(T)	D(T)	E(T)	G(T)	H(T)	M(T)
0.1	0.001	0.006	0.018	0.018	0.018	0.096	0.014	0.018
0.2	0.004	0.022	0.066	0.063	0.063	0.185	0.051	0.062
0.3	0.013	0.047	0.135	0.128	0.126	0.270	0.105	0.125
0.4	0.029	0.078	0.220	0.206	0.203	0.352	0.172	0.200
0.5	0.052	0.116	0.316	0.295	0.289	0.434	0.250	0.284
0.6	0.082	0.158	0.419	0.391	0.381	0.516	0.335	0.373
0.7	0.119	0.205	0.527	0.492	0.477	0.598	0.425	0.465
0.8	0.163	0.257	0.638	0.597	0.576	0.681	0.520	0.560
0.9	0.214	0.312	0.751	0.705	0.677	0.766	0.617	0.657
1.0	0.271	0.371	0.865	0.814	0.779	0.852	0.716	0.755
2.0	1.055	1.116	1.963	1.922	1.807	1.773	1.736	1.750
5.0	4.000	4.001	5.000	4.999	4.821	4.750	4.750	4.750
8.0	7.000	7.000	8.000	8.000	7.822	7.750	7.750	7.750
10.0	9.000	9.000	10.000	10.000	9.822	9.750	9.750	9.750

TABLE II. M(t) and approximations for the gamma distribution with mean 1 and variance 1/2.

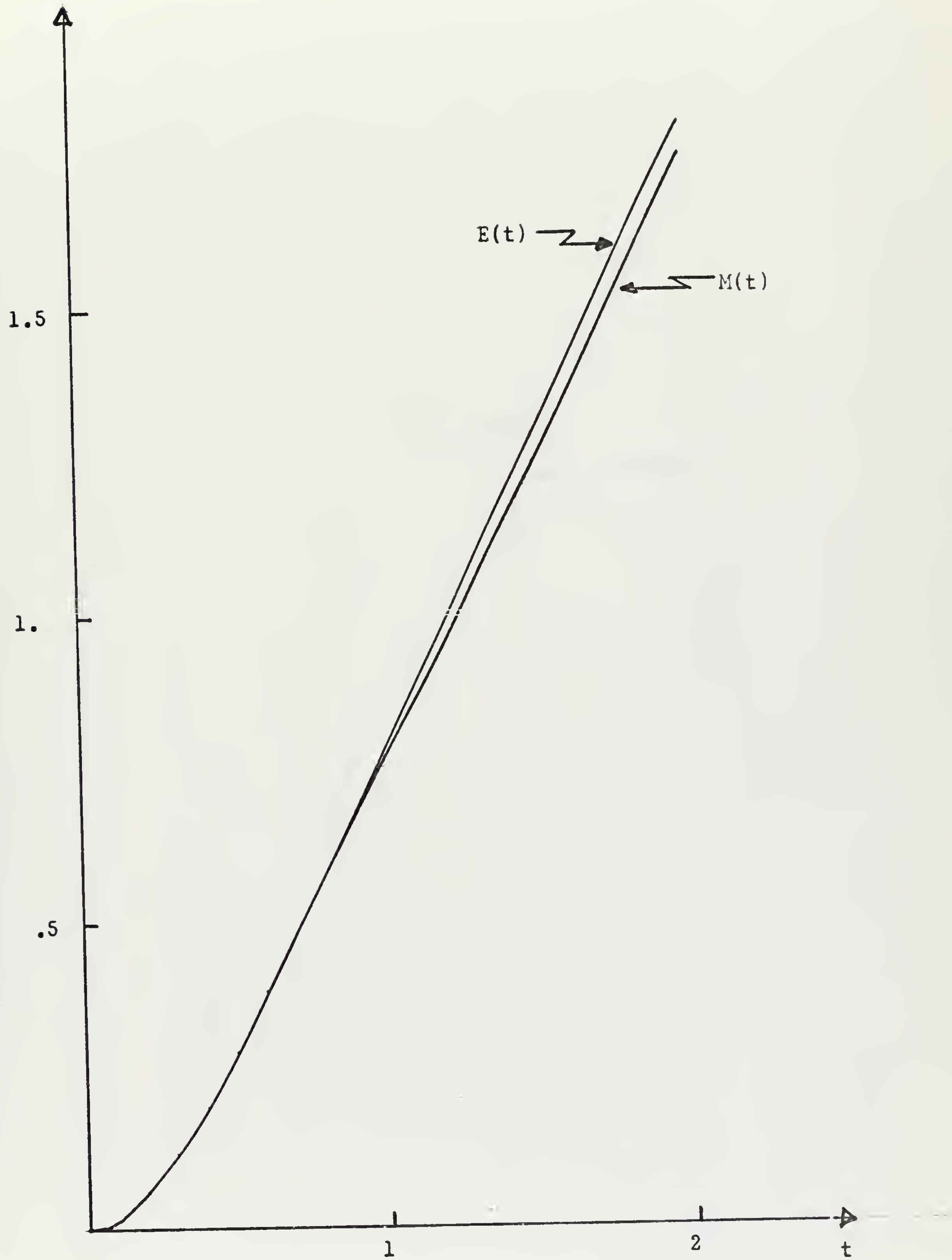

FIGURE 4 : M(t) and E(t) for the gamma distribution.

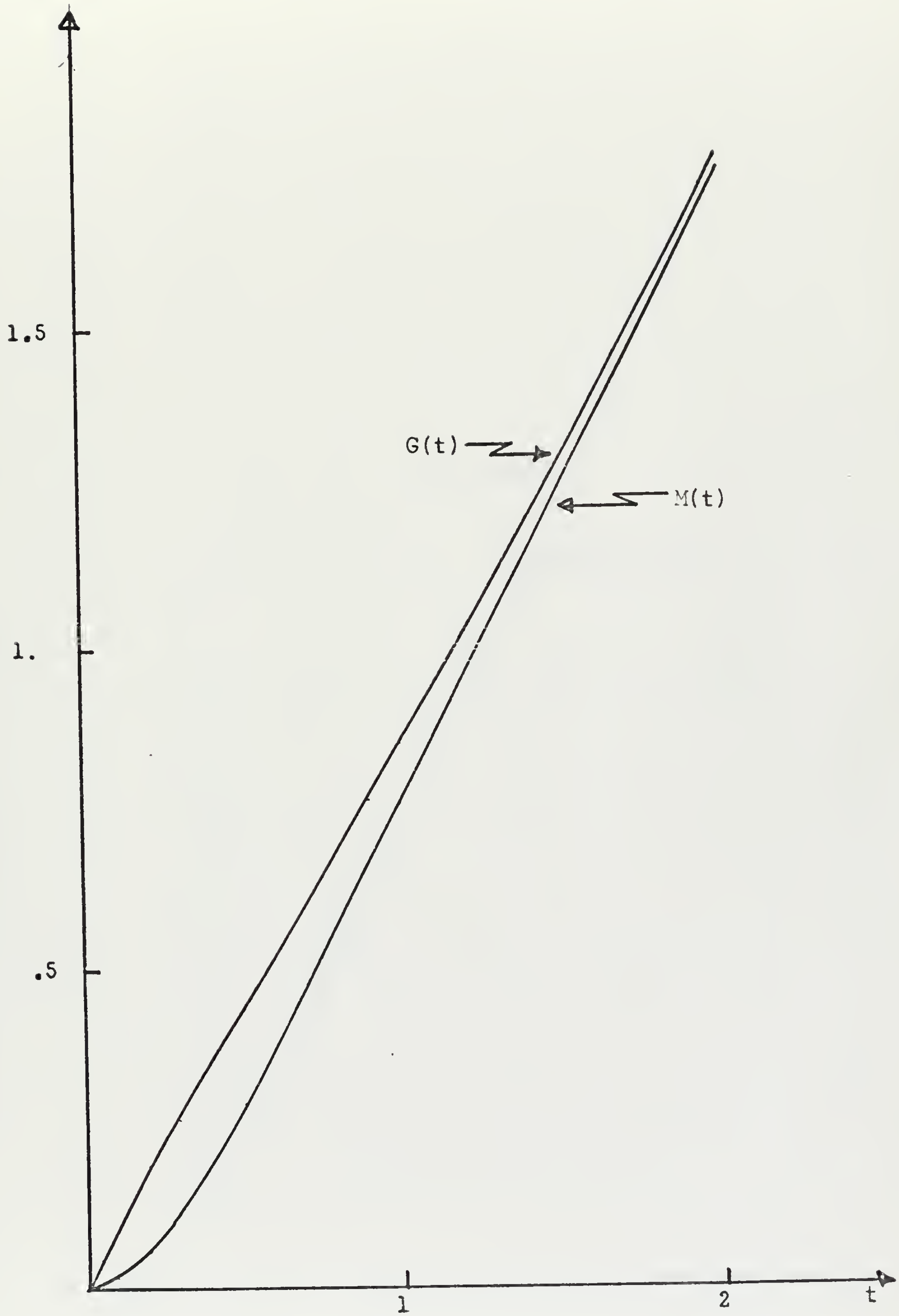

FIGURE 5 : M(t) and G(t) for the gamma distribution.

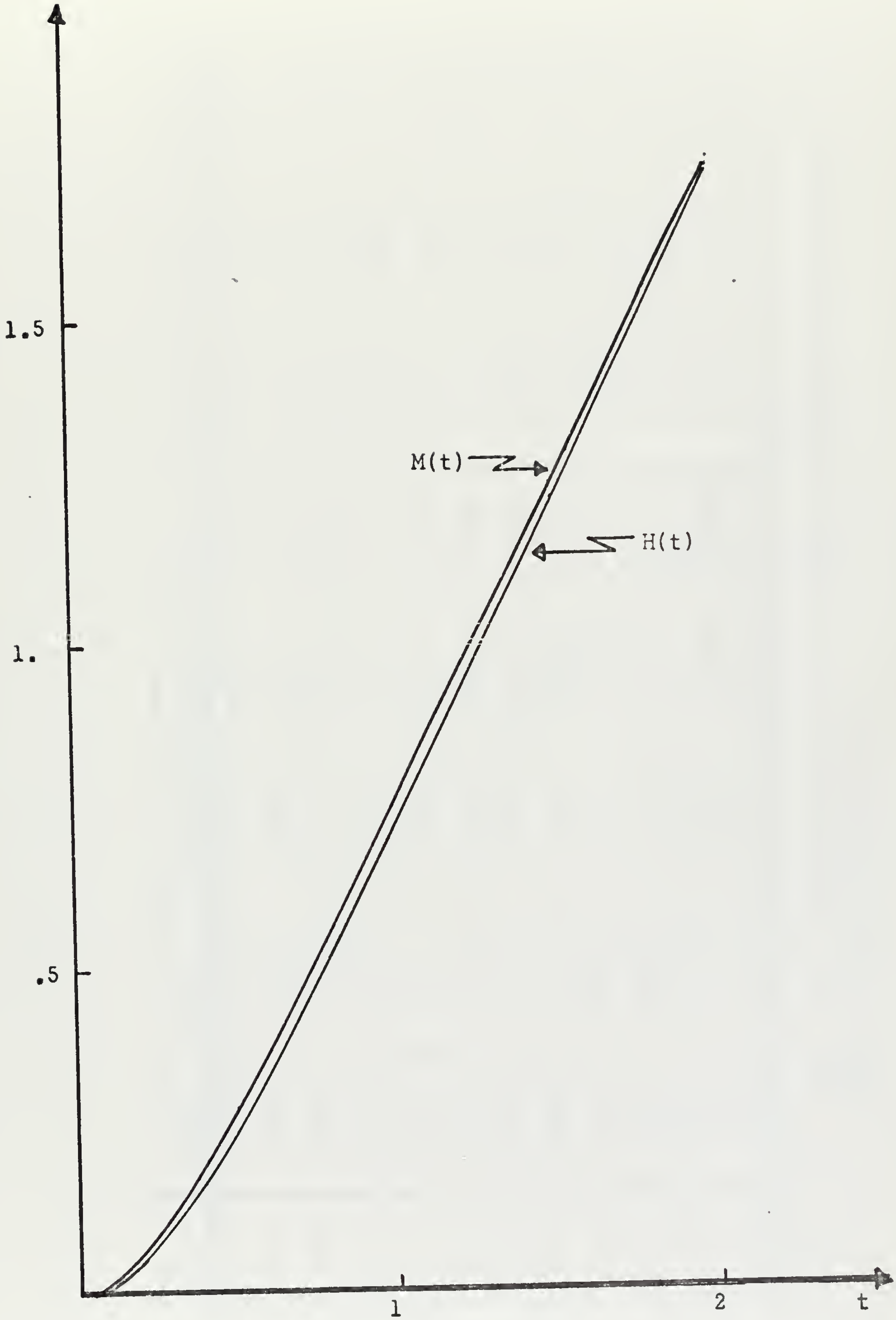

FIGURE 6 : M(t) and H(t) for the gamma distribution.

T	A(T)	B(T)	C(T)	D(T)	E(T)	G(T)	H(T)	M(T)
0.1	0.008	0.081	0.153	0.158	0.158	0.103	0.263	0.158
0.2	0.028	0.165	0.295	0.310	0.311	0.210	0.495	0.311
0.3	0.060	0.251	0.427	0.459	0.460	0.321	0.702	0.460
0.4	0.101	0.338	0.551	0.602	0.605	0.435	0.889	0.605
0.5	0.150	0.427	0.669	0.741	0.747	0.551	1.059	0.747
0.6	0.205	0.517	0.782	0.876	0.886	0.669	1.215	0.886
0.7	0.265	0.609	0.891	1.007	1.022	0.788	1.361	1.022
0.8	0.330	0.701	0.997	1.134	1.156	0.907	1.497	1.155
0.9	0.398	0.793	1.100	1.258	1.287	1.027	1.626	1.285
1.0	0.470	0.886	1.201	1.378	1.415	1.147	1.749	1.413
2.0	1.288	1.808	2.162	2.454	2.617	2.332	2.817	2.599
5.0	4.085	4.462	5.051	5.277	5.837	5.623	5.775	5.736
8.0	7.025	7.209	8.015	8.125	8.893	8.712	8.758	8.749
10.0	9.011	9.116	10.007	10.069	10.906	10.733	10.753	10.750

TABLE III. M(t) and approximations for the hyperexponential distribution with mean 1 and variance 5/2.

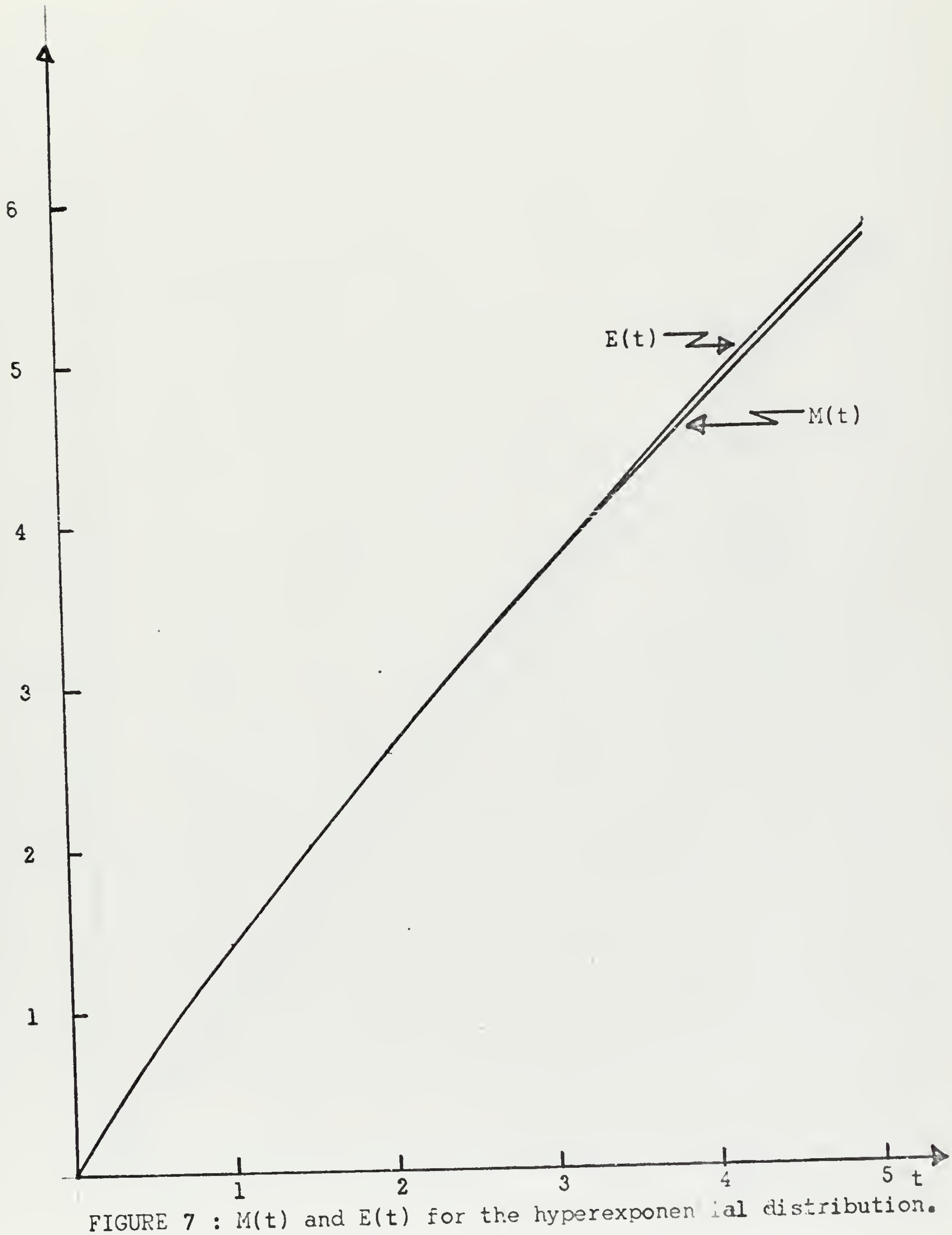

FIGURE 7 : M(t) and E(t) for the hyperexponen ial distribution.

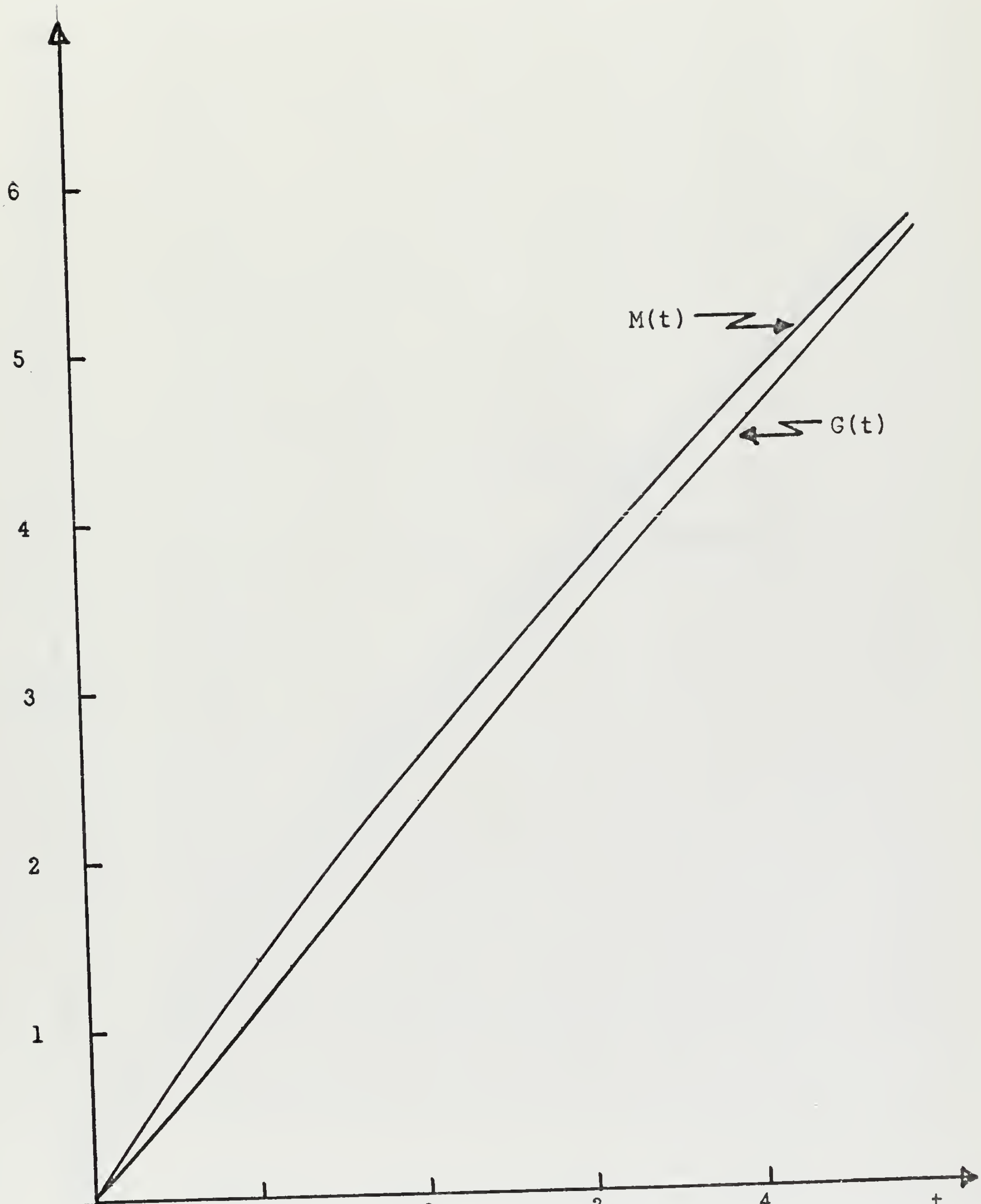

FIGURE 8 : M(t) and G(t) for the hyperexponential distribution.

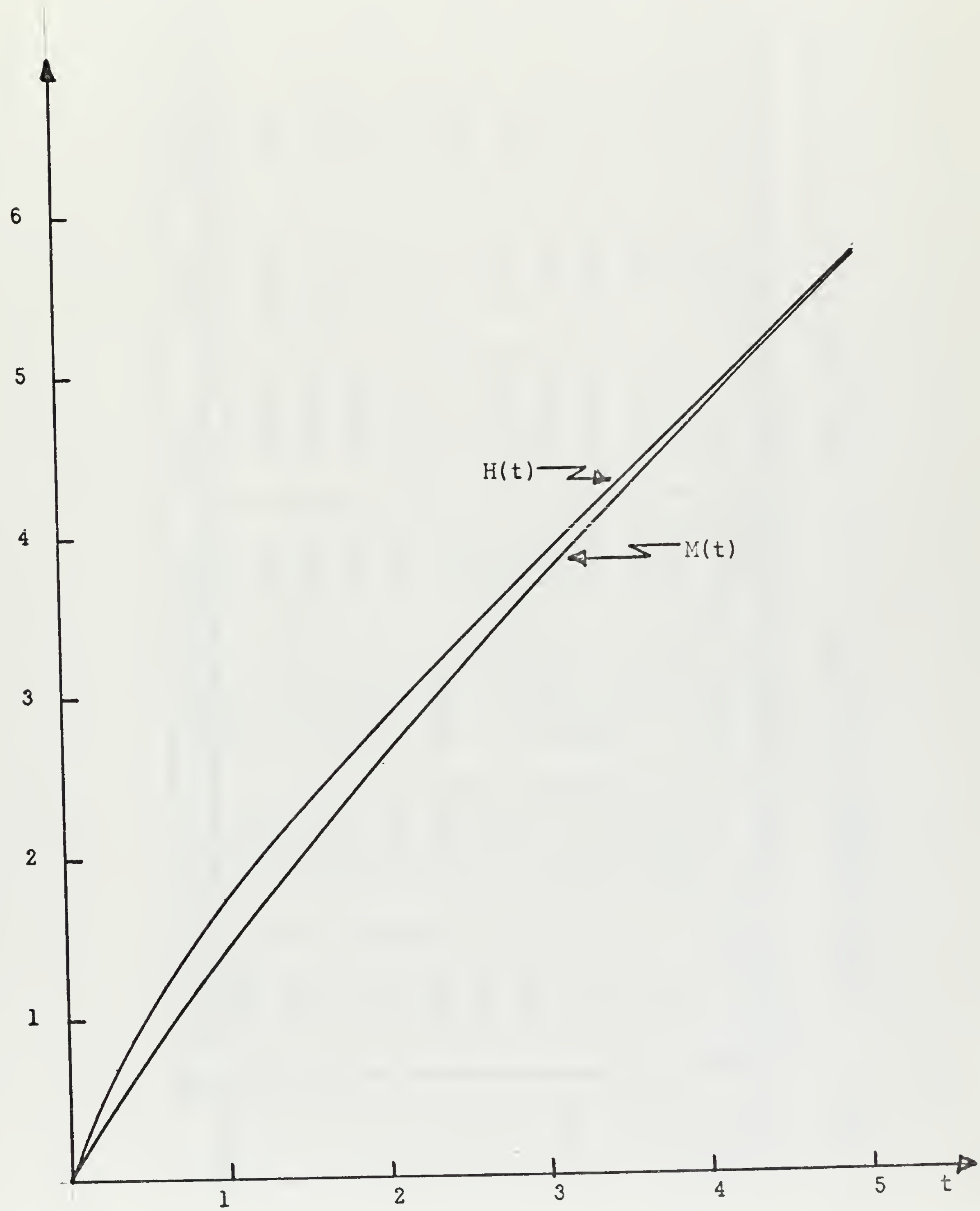

FIGURE 9 : M(t) and H(t) for the hyperexponential distribution.

T	A(T)	B(T)	C(T)	D(T)	E(T)	G(T)	H(T)	M(T)
0.1	0.000	0.003	0.011	0.011	0.011	0.059	0.015	0.007
0.2	0.002	0.017	0.056	0.055	0.054	0.116	0.075	0.056
0.3	0.007	0.040	0.121	0.119	0.118	0.173	0.162	0.122
0.4	0.016	0.070	0.196	0.192	0.190	0.230	0.260	0.179
0.5	0.029	0.105	0.273	0.270	0.265	0.288	0.361	0.265
0.6	0.045	0.143	0.350	0.348	0.341	0.348	0.460	0.344
0.7	0.066	0.183	0.426	0.427	0.417	0.408	0.556	0.412
0.8	0.089	0.225	0.501	0.504	0.492	0.469	0.649	0.479
0.9	0.116	0.268	0.574	0.581	0.566	0.531	0.738	0.545
1.0	0.145	0.313	0.645	0.657	0.639	0.594	0.824	0.612
5.0	2.141	2.400	3.087	3.217	3.267	3.189	3.427	3.189
15.0	8.111	8.218	9.108	9.187	9.441	9.403	9.466	9.305
25.0	14.167	14.216	15.166	15.207	15.533	15.501	15.525	15.440
35.0	20.230	20.256	21.230	21.252	21.643	21.578	21.589	21.539

TABLE IV. M(t) and approximations for the log-normal distribution with mean $e^{\frac{1}{2}}$ and variance $(e^2 - e)$.

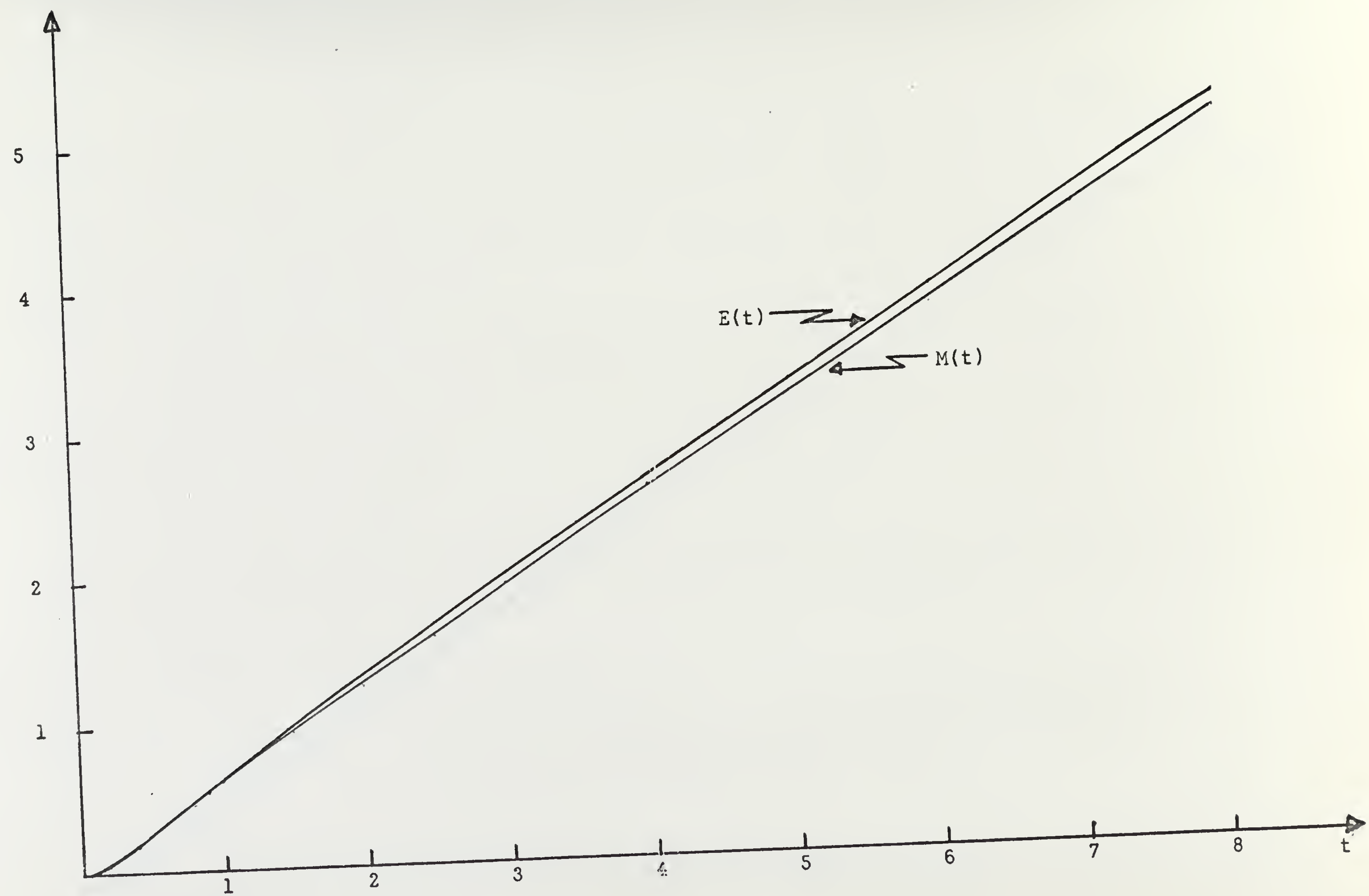

FIGURE 10 : M(t) and E(t) for the log-normal distribution.

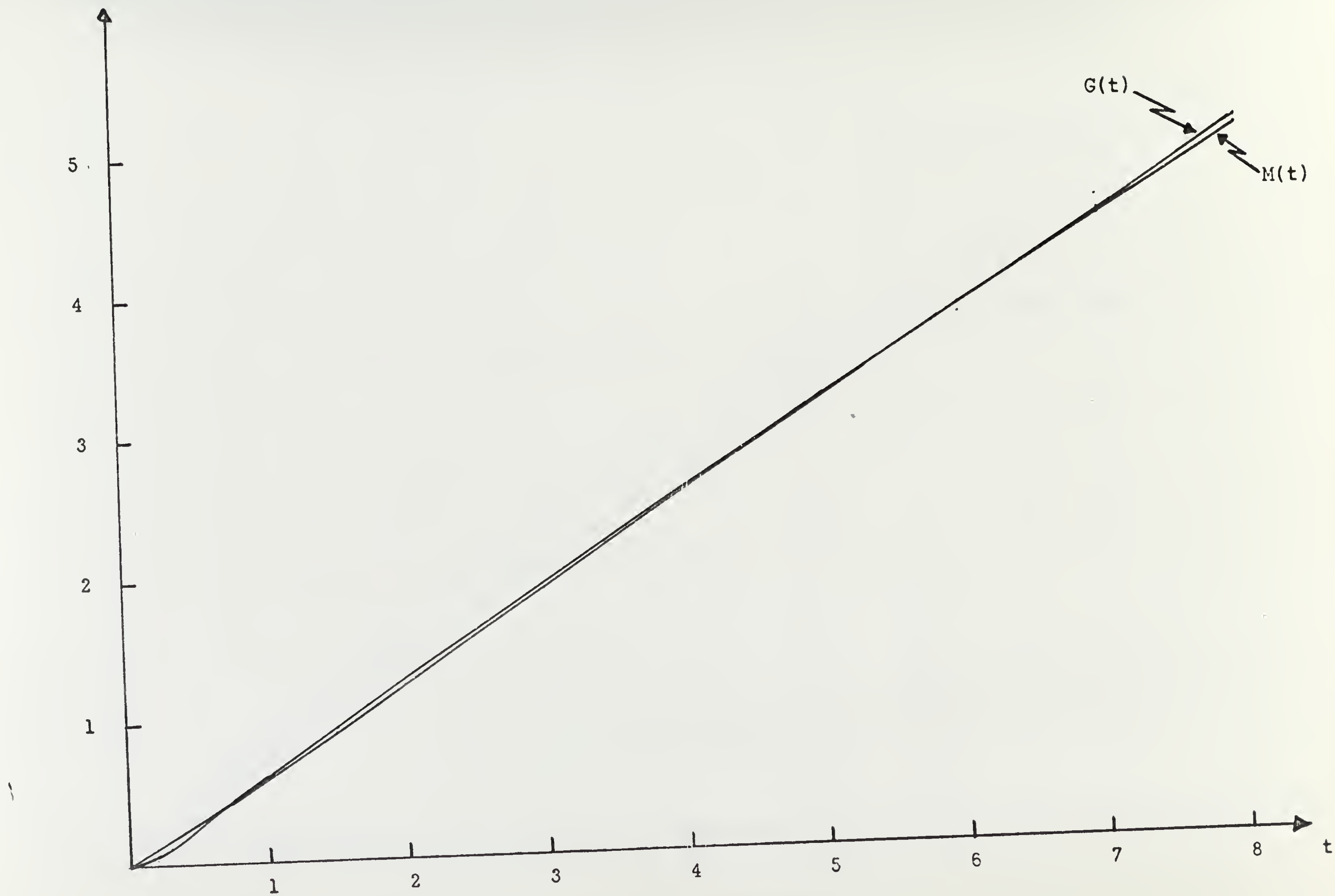

FIGURE 11 : M(t) and G(t) for the log-normal distribution.

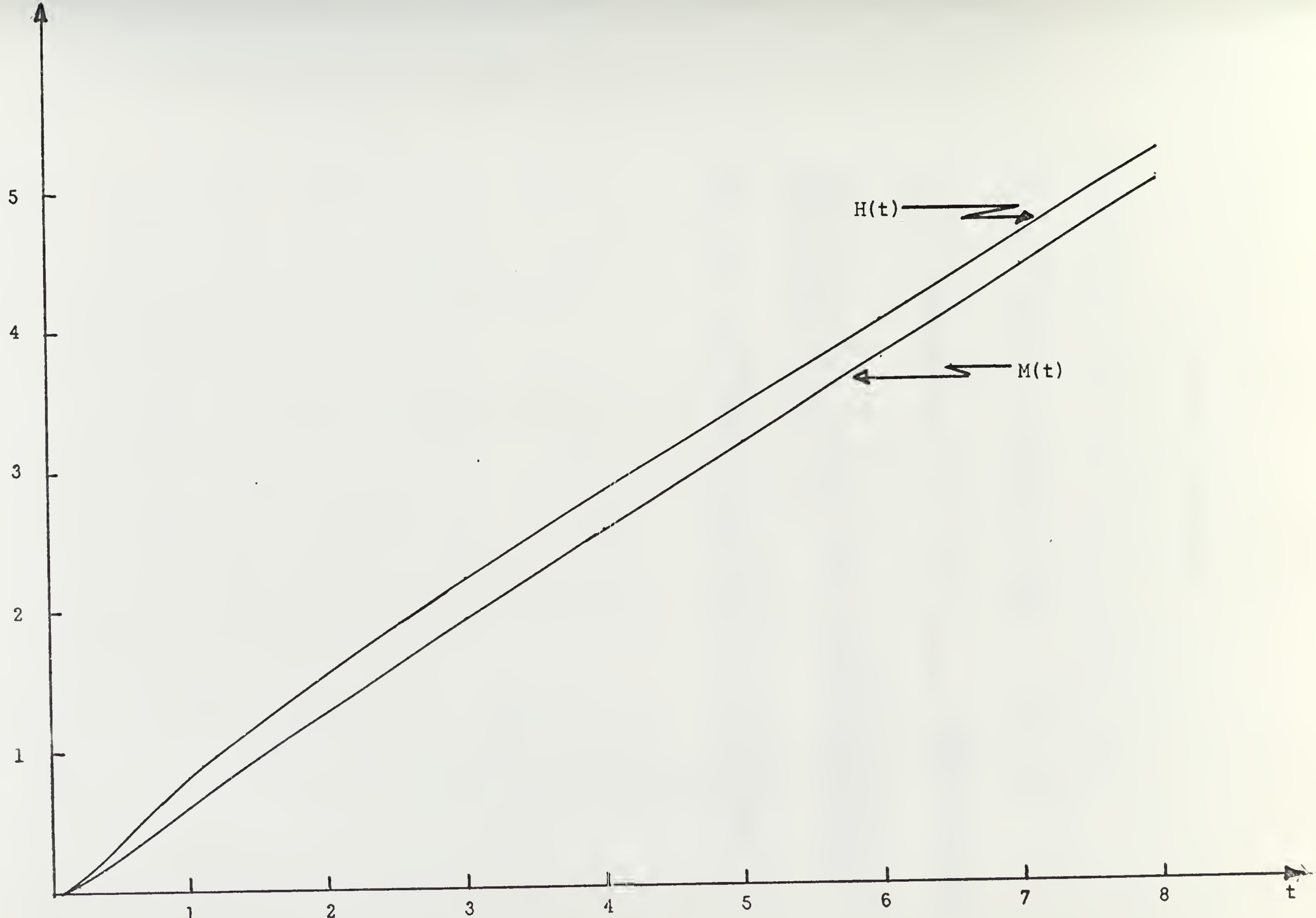

FIGURE 12 : M(t) and H(t) for the log-normal distribution.

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13. ABSTRACT

This thesis gives a number of approximations and bounds for the renewal function in an ordinary renewal process. Each approximation and bound is calculated for the uniform, gamma and hyperexponential distributions and compared with the renewal function for these cases. They are also calculated for the log-normal distribution and compared with results of the simulation of the renewal function. Results are tabulated and shown graphically.

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